Universität Regensburg
Fakultät für Mathematik

Higher category theory and homotopical algebra
Winter Semester 2016/2017

Exercise Sheet 3

1 Localization of an ∞-category

Let $\mathcal{C}$ be an ∞-category and let S be a subcategory of $\mathcal{C}$.

Definition 1.1. *For any ∞-category X, denote by $\underline{Hom}_S(\mathcal{C}, X)$ the full subcategory of $\underline{Hom}(\mathcal{C}, X)$ consisting of the maps which send every morphism in S to an invertible morphism in X.*

*A **localization** of $\mathcal{C}$ with respect to S is a pair $(L(\mathcal{C}), \gamma)$, where $L(\mathcal{C})$ is an ∞-category and $\gamma : \mathcal{C} \to L(\mathcal{C})$ such that for any ∞-category X, γ induces an equivalence of ∞-categories*

$$\underline{Hom}(L(\mathcal{C}), X) \xrightarrow{\gamma^*} \underline{Hom}_S(\mathcal{C}, X).$$

Exercise 1.2. Show that if a localization exists, then it is unique up to equivalence of ∞-categories.

In what follows we want to show the existence of a localization. First let us deal with the case where $\mathcal{C} = S$:

Exercise 1.3. Suppose that $\mathcal{C} = S$. Show that $\mathrm{Ex}^\infty(S)$ together with the canonical map $\beta_S : S \to \mathrm{Ex}^\infty(S)$, as constructed in [Cisinski, 11.22.4 and 11.22.5], is a localization of S with respect to all its maps.

Now we deal with the general case. Consider the following pushout diagram:

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Ex}^\infty(S) & \longrightarrow & \mathcal{C}'. \end{array}$$

Let $L(\mathcal{C})$ be a fibrant replacement of the $\mathcal{C}'$ above in the Joyal model structure, and $\gamma : \mathcal{C} \to L(\mathcal{C})$ be the canonical morphism.

Exercise 1.4. Show that $(L(\mathcal{C}), \gamma)$ is a localization of $\mathcal{C}$ with respect to S.

Now we show that the localization of an ∞-category is compatible with the localization of a usual category:

Exercise 1.5. Show that $\tau L(\mathcal{C})$ is equivalent to the localization of $\tau(\mathcal{C})$ with respect to $\tau(S)$.

Corollary 1.6. Let $\mathcal{M}$ be a model category with W the class of weak equivalences. Then $L(\mathcal{M})$ is an ∞-category such that $\tau L(\mathcal{M})$ is equivalent to the homotopy category $\mathrm{Ho}(\mathcal{M})$.

2 On the Künneth formula

For any simplicial set X, denote by $\mathbb{Z}[X]$ the associated chain complex of free abelian groups, which is such that

$$\mathbb{Z}[X]_n = \oplus_{X_n}\mathbb{Z}.$$

Exercise 2.1. Show that $\mathbb{Z}[X]$ is indeed a chain complex by constructing explicitly the boundary maps.

Then we can see that $\mathbb{Z}[\cdot]$ defines a functor

$$\mathbb{Z}[\cdot] : sSet \to C^+(Ab)$$

from the category of simplicial sets to the category of chain complexes of abelian groups concentrated in non-negative degrees.

Exercise 2.2. Let X and Y be two simplicial sets. Show that there is a canonical map $c_{X,Y} : \mathbb{Z}[X] \otimes \mathbb{Z}[Y] \to \mathbb{Z}[X \times Y]$.

The aim of this exercise is to show that for any X, Y, the map $c_{X,Y}$ is a quasi-isomorphism. Start with some explicit computations:

Exercise 2.3. Show that the canonical map $\mathbb{Z}[\Delta^n] \to \mathbb{Z}$ is a quasi-isomorphism.

We know that there is a model structure on $C^+(Ab)$ which is the *projective model structure*: its weak equivalences are quasi-isomorphisms, and cofibrations are degreewise monomorphisms with degreewise projective kernel ([Hovey, 2.3]). We want to show that the functor $\mathbb{Z}[\cdot]$ is a left Quillen functor for the Kan-Quillen model structure on $sSet$ and the projective model structure on $C^+(Ab)$.

Let S be the set of monomorphisms $X \to Y$ in $sSet$ which induce quasi-isomorphisms $\mathbb{Z}[X] \simeq \mathbb{Z}[Y]$.

Exercise 2.4. Show that the set S is saturated.

Exercise 2.5. Show that for any n and any $0 \leqslant k \leqslant n$, the canonical map $\mathbb{Z}[\Lambda^n_k] \to \mathbb{Z}[\Delta^n]$ is a quasi-isomorphism.

Hint: Use an inductive argument to show that $\mathbb{Z}[\Lambda^n_J] \to \mathbb{Z}[\Delta^n]$ *is a quasi-isomorphism for* J *a non-empty subset of* $[0, \cdots, n]$.

Exercise 2.6. Show that the set S contains all trivial cofibrations in $sSet$ and deduce that the functor $\mathbb{Z}[\cdot]$ is a left Quillen functor.

Exercise 2.7. Conclude that $c_{X,Y}$ is an quasi-isomorphism.

Remark 2.8. As a particular case, if X is a topological space, the homology groups of $\mathbb{Z}[\mathrm{Sing}(X)]$ are exactly the classical singular homology groups of X. Therefore the results above give a purely abstract proof of the classical Künneth formula for singular homology.

References

[Cisinski] D.-C. Cisinski, *Higher category theory and homotopical algebra*, Lecture notes, Winter Semester 2016/2017, http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf.

[Hovey] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs **63**, American Mathematical Soc., 1999.